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REALIZATION OF NONAXISYMMETRICAL MOMENT-FREE STATE IN SHELLS OF REVOLUTION

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A series of formulations of problems involving realization of a moment-free stressed state in elastic reinforced shells with arbitrary shape of the center surface is given in [1]. This paper is concerned with solving three of the problems proposed in [1] for the case when the center surface of the shell is a surface of revolution with nonzero Gaussian curvature. The problem of the possibility of realizing a moment-free state in arbitrary reinforced shells with zero curvature was examined in [2] and the particular case of axial symmetry was examined in [3].

1. We shall examine a shell of revolution with a quasiuniform layered structure over the thickness. We shall choose a system of coordinates fixed to the lines of principle curvature of the surface of the shell. If the shell functions in a moment-free stress state, then the following relations must be satisfied [1]:

equations of equilibrium

$$\begin{gathered}\partial(rT_1)/\partial\varphi - T_2R_1\cos\varphi + R_1\partial T_{12}/\partial\theta = -rR_1p_1,\\ R_1\partial T_2/\partial\theta + \partial(rT_{12})/\partial\varphi + T_{12}R_1\cos\varphi = -rR_1p_2,\\ T_1R_2 + T_2R_1 = R_1R_2p_3;\end{gathered} \tag{1.1}$$

elasticity relations

$$\begin{gathered}T_1 = h(a_{11}\varepsilon_1 + a_{12}\varepsilon_2 + a_{13}\varepsilon_{12}),\ T_2 = h(a_{12}\varepsilon_1 + a_{22}\varepsilon_2 + a_{23}\varepsilon_{12}),\\ T_{12} = T_{21} = h(a_{13}\varepsilon_1 + a_{23}\varepsilon_2 + a_{33}\varepsilon_{12});\end{gathered} \tag{1.2}$$

geometric equations

$$\begin{gathered}\varepsilon_1 = \frac{1}{R_1}\frac{\partial u}{\partial\varphi} + \frac{w}{R_1},\quad \varepsilon_2 = \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{\cos\varphi}{r}u + \frac{w}{R_2},\\ \varepsilon_{12} = \frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{r}{R_1}\frac{\partial}{\partial\varphi}\left(\frac{v}{r}\right);\end{gathered} \tag{1.3}$$

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$$\varkappa_1 = \frac{1}{R_1}\frac{\partial \vartheta_1}{\partial \varphi} = 0, \quad \varkappa_2 = \frac{1}{r}\frac{\partial \vartheta_2}{\partial \theta} + \frac{\cos\varphi}{r}\vartheta_1 = 0, \tag{1.4}$$
$$\tau = -\frac{1}{rR_1}\left(\frac{\partial^2 w}{\partial\varphi\partial\theta} - \frac{R_1\cos\varphi}{r}\frac{\partial w}{\partial\theta}\right) + \frac{1}{rR_1}\frac{\partial u}{\partial\theta} + \frac{r}{R_1R_2}\frac{\partial}{\partial\varphi}\left(\frac{v}{r}\right) = 0,$$
$$\vartheta_1 = -\frac{1}{R_1}\frac{\partial w}{\partial\varphi} + \frac{u}{R_1}, \quad \vartheta_2 = -\frac{1}{r}\frac{\partial w}{\partial\theta} + \frac{v}{R_2};$$

equations of continuity of deformations

$$R_1\partial\varepsilon_{12}/\partial\theta - r\partial\varepsilon_2/\partial\varphi - (\varepsilon_2 - \varepsilon_1)R_1\cos\varphi = 0, \tag{1.5}$$
$$r\partial\varepsilon_{12}/\partial\varphi + \varepsilon_{12}(R_1 + R_2)\cos\varphi - R_1\partial\varepsilon_1/\partial\theta = 0,$$

$$\frac{\partial}{\partial\varphi}\left\{\frac{1}{R_1}\left[r\frac{\partial\varepsilon_2}{\partial\varphi} + (\varepsilon_2 - \varepsilon_1)R_1\cos\varphi - \frac{R_1}{R_2}\frac{\partial\varepsilon_{12}}{\partial\theta}\right]\right\} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(R_1\frac{\partial\varepsilon_1}{\partial\theta} - \frac{r}{2}\frac{\partial\varepsilon_{12}}{\partial\varphi} - \varepsilon_{12}R_1\cos\varphi\right) = 0.$$

The generalized forces

$$T_1 = T_\varphi, \; T_{12} = T_{\varphi\theta}, \; T_{\varphi n} = 0, \; M_\varphi = 0 \tag{1.6}$$

or the corresponding generalized displacements

$$u, v, w, \vartheta_1; \tag{1.7}$$

must be given on the boundary of the shell φ = const [4], while the generalized forces

$$T_2 = T_\theta, \; T_{21} = T_{\theta\varphi}, \; T_{\theta n} = 0, \; M_\theta = 0 \tag{1.8}$$

or the generalized displacements

$$u, v, w, \vartheta_2 \tag{1.9}$$

must be given on the boundary θ = const.

In (1.1)-(1.9), T_1, T_2, T_{12} are the forces; ε_1, ε_2, ε_{12}, components of tangential deformation of the center surface; $\varkappa_1$, $\varkappa_2$, τ, components of the bending deformation; u, v, and w, components of displacement; p_1, p_2, p_3, components of external surface load; R_1, R_2, principle radii of curvature; $r = R_2 \sin\varphi$, instantaneous radius; h, thickness of the shell; a_{km}, generalized elastic characteristics of a typical layer of the shell; T_φ, $T_{\varphi\theta}$, $T_{\varphi n}$, M_φ, components of the vector of external forces and bending moment, applied at the boundary φ = const; T_θ, $T_{\theta\varphi}$, $T_{\theta n}$, M_θ, analogous quantities given on the contour θ = const; ϑ_1, ϑ_2, angles of rotation of the normal to the center surface.

2. Equations (1.1)-(1.3) are the equations of classical moment-free theory [5] for the shell under study (principle system of equations of moment-free theory [6]). Together with the conditions for the absence of bending deformation (1.4) and the equations of continuity (1.5), these equations determine the stressed–deformed state of the shell of revolution with finite bending rigidity, functioning in a strictly moment-free regime.

The solution of the equations of continuity (1.5) gives the general form of the field of tangential deformations of the surface of revolution with bending-free deformations of the latter. To find it, we eliminate in (1.5) the functions ε_1, ε_2 and obtain

$$\frac{\partial}{\partial\theta}\left(\frac{\partial\varepsilon_{12}}{\partial\varphi} + \varepsilon_{12}\operatorname{ctg}\varphi\right) = 0.$$

Solving this equation and integrating the first two equations from (1.5), we find

$$\varepsilon_{12} = \xi_1(\theta)\sin^{-1}\varphi + \xi_2(\varphi), \tag{2.1}$$
$$\varepsilon_1 = \xi_3(\varphi) + \operatorname{ctg}\varphi\int_{\theta_0}^{\theta}\xi_1(\theta)\,d\theta + \frac{\theta-\theta_0}{R_1\sin\varphi}\frac{d}{d\varphi}(\xi_2 r\sin\varphi),$$
$$\varepsilon_2 = \frac{1}{r}\left[\xi_4(\theta) + \int_{\varphi_0}^{\varphi}R_1\left(\frac{1}{\sin\varphi}\frac{d\xi_1}{d\theta} + \varepsilon_1\cos\varphi\right)d\varphi\right].$$

Here $\xi_1(\theta)$, $\xi_2(\varphi)$, $\xi_3(\varphi)$, $\xi_4(\theta)$ are functions of integration; the lower limits of integration φ_0, θ_0 are chosen arbitrarily.

We note some properties of the solution obtained, related to the form of the central surface.

1. When the center surface contains a point corresponding to a smooth vertex ($\varphi = 0$ or $\varphi = \pi$), it follows from (2.1) that the functions ε_{12}, ε_1, ε_2 will be finite at this point if we assume (for $\varphi_0 = 0, \pi$)

$$\xi_1(\theta) = 0, \quad \xi_4(\theta) = 0. \tag{2.2}$$

Therefore, for moment-free deformation of such a shell, the magnitude of the shear deformations ε_{12} does not depend on the angle θ, while the relative elongations ε_1, ε_2 in the general case depend linearly on θ.

2. If the shell is closed in the circular direction, then from the condition of periodicity of the functions ε_{12}, ε_1, ε_2 with respect to θ we obtain the dependence

$$\xi_2(\varphi) = \frac{1}{r \sin\varphi}\left(c - \frac{r}{2\pi}\int_0^{2\pi} \xi_1(\theta)\, d\theta\right), \tag{2.3}$$

where c is an arbitrary constant; $\xi_1(\theta)$, $\xi_4(\theta)$ are periodic functions.

3. When the center surface contains a smooth vertex and is closed in the circular direction, we find from (2.1)-(2.3)

$$\varepsilon_{12} = 0, \quad \varepsilon_1 = \xi_3(\varphi) = \varepsilon_1(\varphi), \quad \varepsilon_2 = \frac{1}{r}\int_0^{\varphi} \xi_3 R_1 \cos\varphi d\varphi = \varepsilon_2(\varphi), \tag{2.4}$$

i.e., in such a shell, with bending-free deformation the field of tangential deformations is axisymmetrical and, in addition, the deformations ε_1 and ε_2 are the principle deformations.

When the equations of continuity (1.5) are satisfied, displacements in a strictly moment-free shell are determined from Eqs. (1.3) and (1.4) in the form

$$u = A\cos\varphi + C\sin\varphi + R_1\vartheta_1, \quad w = A\sin\varphi, \quad v = \frac{\partial A}{\partial\theta} - R_2\cos\varphi\int_{\theta_0}^{\theta} \vartheta_1 d\theta + \eta_3(\varphi), \tag{2.5}$$

where

$$A = B + \eta_2(\theta); \quad B = \int_{\varphi_0}^{\varphi} C d\varphi; \quad C = [(J_1 + \eta_1(\varphi)) R_2 + (R_2 - R_1)\vartheta_1]\frac{1}{\sin\varphi};$$

$$\eta_1(\varphi) = \frac{1}{r}\left[c_1 + r_0 J_1(\varphi_0, \theta) + \int_{\varphi_0}^{\varphi} \varepsilon_1 R_1 \sin\varphi d\varphi\right] - J_1; \quad J_1 = \sin\varphi\int_{\theta_0}^{\theta} \varepsilon_{12} d\theta;$$

$$\eta_2(\theta) = (c_2 + J_2)\sin\theta + (c_3 + J_3)\cos\theta; \quad \vartheta_1 = (c_4 + J_4)\sin\theta + (c_5 + J_5)\cos\theta;$$

$$\eta_3(\varphi) = r\left[c_6 + \left(\frac{d\vartheta_1}{d\theta} + \int_{\theta_0}^{\theta} \vartheta_1 d\theta\right)\operatorname{ctg}\varphi - \int_{\varphi_0}^{\varphi} \frac{\varepsilon_{12}}{\sin\varphi} d\varphi\right];$$

$$J_2 = \int_{\theta_0}^{\theta} \eta\cos d\theta; \quad J_3 = -\int_{\theta_0}^{\theta} \eta\sin\theta d\theta; \quad \eta = r\varepsilon_2 - (J_1 + \eta_1) R_2\cos\varphi - \frac{\partial^2 B}{\partial\theta^2} - B;$$

$$J_4 = -\sin\varphi\int_{\theta_0}^{\theta} \frac{\partial\varepsilon_{12}}{\partial\theta}\cos\theta d\theta; \quad J_5 = \sin\varphi\int_{\theta_0}^{\theta} \frac{\partial\varepsilon_{12}}{\partial\theta}\sin\theta d\theta.$$

Here $c_1, \ldots, c_6$ are arbitrary constants. Thus, in contrast to the classical moment-free theory [4, 6], the geometric equations of a shell functioning in a strictly moment-free stress state can be solved explicitly in a general form. This circumstance is a result of the possibility of constructing an explicit solution of the geometric equations of the general moment theory [6].

When the shell has a smooth vertex ($\varphi = 0$), is closed in the circular direction (dome), and is completely clamped along the bounding contour $\varphi = \varphi_1$ against tangential displacements

$$u(\varphi_1, \theta) = 0, \quad v(\varphi_1, \theta) = 0,$$

we obtain from (2.5)

$$u = \varepsilon_2 r\cos\varphi + \sin\varphi\left(c_1 + \int_{\varphi_1}^{\varphi} \varepsilon_1 R_1 \sin\varphi d\varphi\right), \tag{2.6}$$

$$w = \varepsilon_2 r \sin\varphi - \cos\varphi \left(c_1 + \int_{\varphi_1}^{\varphi} \varepsilon_1 R_1 \sin\varphi d\varphi \right), \tag{2.6}$$

$$v = 0,\ \vartheta_1 = 0,\ c_1 = \varepsilon_2(\varphi_1) R_2(\varphi_1) \cos\varphi_1.$$

3. We shall formulate and examine further three problems involving the realization of a moment-free stress state in reinforced shells of revolution.

Problem 1. Assume that the following are given for a shell of revolution: shape of the center surface, laws of variation of the thickness and nature of the anisotropy, boundary conditions of the type (1.6)-(1.9). The problem is to determine the components p_1, p_2, p_3 of the surface load that gives rise to a moment-free state in the shell.

Substituting the expressions for ε_1, ε_2, ε_{12} from (2.1) into the relations of elasticity (1.2) and further into the equation of equilibrium (1.1), we obtain an expression for the components of the surface load sought in terms of the quantities and functions $\xi_1(\theta)$, $\xi_2(\varphi)$, $\xi_3(\varphi)$, $\xi_4(\theta)$ known from the conditions of the problem. Thus the solution of problem 1 reduces to finding the functions ξ_1, ξ_2, ξ_3, ξ_4 from the boundary conditions. After these functions are found, the forces are determined from Eqs. (1.2), while the displacements are determined from Eqs. (2.5).

If it is not possible to satisfy all boundary conditions stated due to the functions ξ_1, ξ_2, ξ_3, ξ_4, then this means that in this shell it is impossible to realize a moment-free state by choosing the surface load.

As an example, we shall examine the solution of problem 1 for a closed shell of revolution with smooth vertices under the condition that along the meridian $\theta = \theta_0$ the value of the normal component of the surface load is given:

$$p_3 |_{\theta=\theta_0} = p(\varphi). \tag{3.1}$$

On the strength of the fact that the shell is closed, it is necessary to require that all of the functions sought be periodic with respect to θ and finite at the points $\varphi = 0$, $\varphi = \pi$.

In the moment-free shell under study, the general form of the field of deformations is determined by relations (2.4). Therefore, to solve the problem it remains to determine the function $\xi_3(\varphi)$ or, which is more convenient, the function ε_2. For this, taking into account (2.4), we represent the last equation in (1.1) with the help of the relations of elasticity (1.2) in the form

$$h(D_1\varepsilon_1 + D_2\varepsilon_2) = p_3, \tag{3.2}$$

where $D_1 = a_{11}R_1^{-1} + a_{12}R_2^{-1}$; $D_2 = a_{12}R_1^{-1} + a_{22}R_2^{-1}$.

If the geometry of the center surface and the nature of the anisotropy of the shell material are such that $D_1 \neq 0$, then from (3.2), taking into account (2.4) and condition (3.1), we obtain

$$\varepsilon_1 = k_1 p_3 - k_2\varepsilon_2 = k_1^0 p - k_2^0 \varepsilon_2, \qquad k_1 = (hD_1)^{-1},\ k_2 = D_2 D_1^{-1}, \tag{3.3}$$

$$k_1^0 = k_1(\varphi_1, \theta_0),\ k_2^0 = k_2(\varphi, \theta_0).$$

It also follows from (2.4) that

$$d(r\varepsilon_2)/d\varphi = \varepsilon_1 R_1 \cos\varphi. \tag{3.4}$$

Substituting here the expression for ε_1 from (3.3) and integrating the expression obtained relative to ε_2, we obtain

$$\varepsilon_2 = \frac{1}{r} \exp(-J) \int_0^{\varphi} p k_1^0 R_1 \cos\varphi \exp J d\varphi,\ J = \int_0^{\varphi} \frac{1}{r} k_2^0 R_1 \cos\varphi\, d\varphi. \tag{3.5}$$

If $D_1 = 0$, then $D_2 \neq 0$, since otherwise the condition that the potential energy of the deformation of the shell be positive definite breaks down. In this case, we immediately determine from (3.1) and (3.2)

$$\varepsilon_2 = p_3 (hD_2)^{-1} = p(hD_2)^{-1} |_{\theta=\theta_0}.$$

We shall illustrate the results obtained when the shell is isotropic and the center surface is closed ellipsoid of revolution. In this case, relations (1.2) with conditions (2.4) have the form

$$T_1 = hE(1 - \nu^2)^{-1}(\varepsilon_1 + \nu\varepsilon_2),\ T_2 = hE(1 - \nu^2)^{-1}(\nu\varepsilon_1 + \varepsilon_2),\ T_{12} = 0.$$

Substituting these expressions into the equation of equilibrium (1.1) and including (2.4), we obtain

$$p_1 = \frac{E}{1-\nu^2}\left[-\left(\frac{1}{R_1}\frac{\partial h}{\partial \varphi} + \frac{1-\nu}{R_2} h \operatorname{ctg}\varphi\right)\varepsilon_1 + \left(\frac{1-\nu}{R_2} h \operatorname{ctg}\varphi - \frac{\nu}{R_1}\frac{\partial h}{\partial \varphi}\right)\varepsilon_2 - \frac{h}{R_1}\left(\frac{\partial \varepsilon_1}{\partial \varphi} + \nu \frac{\partial \varepsilon_2}{\partial \varphi}\right)\right], \tag{3.6}$$

$$p_2 = -\frac{E(\nu\varepsilon_1 + \varepsilon_2)}{(1-\nu^2) r}\frac{\partial h}{\partial \theta}, \quad p_3 = \frac{hE}{1-\nu^2}\left[\left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)\varepsilon_1 + \left(\frac{\nu}{R_1} + \frac{1}{R_2}\right)\varepsilon_2\right].$$

Let the thickness of the shell vary according to the law

$$H = 1 + H_* \sin^2\theta \sin^2\varphi, \tag{3.7}$$

where $H = h/h_0$; $H_* = (h_1 - h_0)/h_0$; $h_0 = h(0, \theta)$; $h_1 = h(\pi/2, \pi/2)$. We shall also assume that

$$p(\varphi) = p = \text{const}. \tag{3.8}$$

For the radii of curvature we have [4]

$$R_1 = R(1 + \gamma \sin^2\varphi)^{-3/2}, \quad R_2 = R(1 + \gamma \sin^2\varphi)^{-1/2}, \quad R = a_2^2 a_1^{-1}, \tag{3.9}$$
$$\gamma = \varepsilon^2 - 1, \quad \varepsilon = a_2 a_1^{-1},$$

where a_1, a_2 are the semiaxes of the ellipsoid (the semiaxis a_1 is situated on the axis of revolution). Since for an isotropic ellipsoid

$$D_1 = (E/(1-\nu^2))(1/R_1 + \nu/R_2) > 0,$$

its deformation is determined by relations (3.3) and (3.4) in which

$$k_1^0 = ((1-\nu^2)/h_0 E)(1/R_1 + \nu/R_2)^{-1}, \; k_2^0 = k_2 = (\nu/R_1 + 1/R_2)(1/R_1 + \nu/R_2)^{-1}. \tag{3.10}$$

Figure 1a-c shows the dependence of the dimensionless quantities $q_1 = p_1/p$, $q_2 = p_2/p$, $q_3 = p_3/p$ on φ, θ, calculated according to Eqs. (3.3) and (3.5)-(3.10) with $\nu = 0.3$, $c = 0.25$, $\theta_0 = 0$, $0 \le \varphi \le \pi/2$, $0 \le \theta \le \pi/2$. It follows from (3.6)-(3.9) that for other values of φ, θ the quantities q_1, q_2, q_3 are determined according to the equalities

$$q_1(\varphi, \theta) = -q_1(\pi - \varphi, \theta) = q_1(\varphi, -\theta) = q_1(\varphi, \pi - \theta),$$
$$q_2(\varphi, \theta) = q_2(\pi - \varphi, \theta) = -q_2(\varphi, -\theta) = -q_2(\varphi, \pi - \theta),$$
$$q_3(\varphi, \theta) = q_3(\pi - \varphi, \theta) = q_3(\varphi, -\theta) = q_3(\varphi, \pi - \theta).$$

The continuous and dashed curves in Figs. 1a-c correspond to values $H_* = 1, 2$; curves 2-5 correspond to the values $\theta = \pi/8, \pi/4, (3/8)\pi, \pi/2$. Curve 1 corresponds to the case $\theta = 0$ (for any H_*).

For $H_* = 0$ the ellipsoid has a constant thickness. Then it follows from (3.6) and (3.7) with the condition (3.8) that $q_1 = q_1(\varphi)$, $q_2 = 0$, $q_3 = 1$. The function $q_1(\varphi)$ with $\varepsilon = 0.25$ is illustrated by the dot–dash curve in Fig. 1a. For a sphere ($\varepsilon = 1$), we obtain from (3.3)-(3.7) that $q_1 = q_2 = 0$, $q_3 = 1$.

4. To describe the elastic properties of the shell material, we shall use the model of a reinforced layer [7]. In this case, the coefficients in (1.2) have the form

$$a_{ii} = \frac{aE}{1-\nu^2} + \sum_{n=1}^{N} \omega_n E_n l_{in}^4, \quad a_{i3} = \sum_{n=1}^{N} \omega_n E_n l_{in}^3 l_{jn}, \tag{4.1}$$

$$a_{12} = \frac{aE\nu}{1-\nu^2} + \sum_{n=1}^{N} \omega_n E_n l_{1n}^2 l_{2n}^2, \quad a_{33} = \frac{aE}{2(1+\nu)} + \sum_{n=1}^{N} \omega_n E_n l_{1n}^2 l_{2n}^2,$$
$$l_{1n} = \cos\psi_n, \quad l_{2n} = \sin\psi_n, \quad i, j = 1, 2, \; i \ne j,$$

where N is the number of families of filaments (reinforcement) oriented in the same direction in a characteristic layer; n is the number of families; ω_n is the relative volume content of filaments of this family; E_n is Young's modulus for the filaments; ψ_n is the angle between the orientation of the filaments in the family and the meridian; E, ν are Young's modulus and Poisson's coefficient of the binding material; a is the relative volume content of the binder in the layer.

Problem 2. Given: the shape of the center surface, the external surface load, the rigidity $G_k = \omega_k E_k (aE)^{-1}$ (preliminary reinforcement, $k = 3, \ldots, N$), the angles $\psi_n(\varphi, \theta)$ ($n = 1, \ldots, N$), and boundary conditions of the form (1.6)-(1.9). On separate sections or lines of the center surface, the values of h, G_1, G_2 are given. The problem is to find the changes in the thickness h and rigidities G_1, G_2 (additional reinforcement) with which a moment-free stressed state is realized in the shell.

In solving problem 2, we shall assume that in each specific case the general solution of the equations of

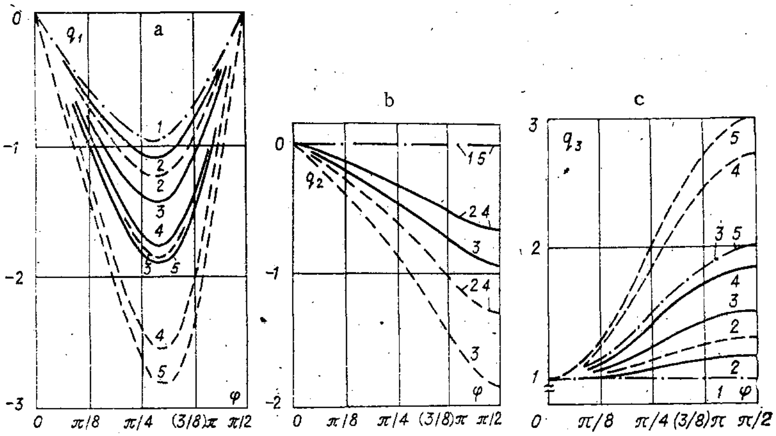

Fig. 1

equilibrium (1.1) is known. (The problem of constructing the integrals of these equations has been investigated quite completely in the moment-free theory [4-6].) Then, respresenting the elasticity relations (1.2) with the help of the dependences (4.1) in the form

$$G_1e_1l_{11}^2 + G_2e_2l_{12}^2 - T_1(aEh)^{-1} = -a'_{11}\varepsilon_1 - a'_{12}\varepsilon_2 - a'_{13}\varepsilon_3,$$
$$G_1e_1l_{21}^2 + G_2e_2l_{22}^2 - T_2(aEh)^{-1} = -a'_{12}\varepsilon_1 - a'_{22}\varepsilon_2 - a'_{23}\varepsilon_3,$$
$$G_1e_1l_{11}l_{21} + G_2l_{12}e_2l_{22} - T_{12}(aEh)^{-1} = -a'_{13}\varepsilon_1 - a'_{23}\varepsilon_2 - a'_{33}\varepsilon_3, \tag{4.2}$$

where

$$a'_{11} = \frac{1}{1-\nu^2} + \sum_{k=3}^{N} G_k l_{1k}^4; \quad a'_{12} = \frac{\nu}{1-\nu^2} + \sum_{k=3}^{N} G_k l_{1k}^2 l_{2k}^2;$$
$$a'_{i3} = \sum_{k=3}^{N} G_k l_{ik}^3 l_{jk}; \quad a'_{33} = \frac{1}{2(1+\nu)} + \sum_{k=3}^{N} G_k l_{1k}^2 l_{2k}^2;$$
$$e_i = \varepsilon_1 l_{1i}^2 + \varepsilon_2 l_{2i}^2 + \varepsilon_3 l_{1i} l_{2i}; \quad \varepsilon_3 = \varepsilon_{12},$$

and assuming that

$$\Delta = [T_1 l_{21} l_{22} + T_2 l_{11} l_{12} - T_{12} \sin(\psi_2 + \psi_1)] \sin(\psi_2 - \psi_1) \neq 0,$$

we find

$$h = \Delta(aE\Delta_3)^{-1}, \; G_i = \Delta_i(e_i\Delta)^{-1}, \; i, j = 1, 2, \; i \neq j. \tag{4.3}$$

Here

$$\Delta_i = (-1)^i \sum_{k=1}^{3} \beta_{ik}\varepsilon_k; \quad \beta_{ik} = a'_{1k}B_i - a'_{2k}C_i + a'_{3k}D_i;$$
$$\Delta_3 = \sin(\psi_2 - \psi_1) \sum_{k=1}^{3} [a'_{1k} l_{21} l_{22} + a'_{2k} l_{11} l_{12} - a'_{3k} \sin(\psi_1 + \psi_2)] \varepsilon_k;$$
$$B_i = (T_2 l_{1j} - T_{12} l_{2j}) l_{2j}; \quad C_i = (T_1 l_{2j} - T_{12} l_{1j}) l_{1j}; \quad D_i = T_1 l_{2j}^2 - T_2 l_{1j}^2.$$

Substituting into (4.3) the values for ε_1, ε_2, ε_3 from (2.1), we obtain expressions for G_1, G_2, h in terms of the given and arbitrary functions, entering into the general integrals T_1, T_2, T_{12}, ε_1, ε_2, ε_3. Therefore, the solution of problem 2 reduces to determining these arbitrary functions from the boundary conditions and the conditions on G_1, G_2, h. Finally, the displacements are found from Eqs. (2.5).

We shall examine the case when the shell is closed in the circular direction, contains a smooth vertex, and the value of the relative rigidity G_2 is given on the contour $\theta = \theta_0$:

$$G_2(\varphi, \theta_0) = g(\varphi). \tag{4.4}$$

In this case, according to the dependences (2.4), the deformation field is determined to within the function

$\varepsilon_2(\varphi)$. To find it, we shall take into account condition (4.4) and represent the third equality in (4.3) in the form

$$\varepsilon_1 = -k\varepsilon_2,\ k = [(\beta_{22} - g_\Delta \sin^2\psi_2)(\beta_{21} - g_\Delta \cos^2\psi_2)^{-1}]_{\theta=\theta_0}. \quad (4.5)$$

Solving Eqs. (4.5) and (3.4) simultaneously, we find

$$\varepsilon_2 = c\exp J,\quad J = -\int_0^\varphi (1+k)\frac{R_2}{R_1}\operatorname{ctg}\varphi\, d\varphi, \quad (4.6)$$

where c is a constant of integration.

As an example, we shall examine the solution of problem 2 for a spherical dome with a vertex ($R_1 = R_2 = R = \text{const}$). On the reference contour $\varphi = \varphi_1$, the dome is fixed so that the following conditions are satisfied

$$u = 0,\ v = 0,\ T_{\varphi n} = 0,\ M_\varphi = 0 \quad \text{for } \varphi = \varphi_1. \quad (4.7)$$

The surface load acting on the shell has the components:

$$p_1 = q\sin\varphi,\ p_2 = 0,\ p_3 = -q\cos\varphi - p\sin\varphi\cos\theta,\ p = \text{const}, \quad (4.8)$$
$$q(\varphi) = q_0 + (q_* - q_0)(1-\cos\varphi)^m(1-\cos\varphi_*)^{-m},\quad m = \text{const},$$

i.e., a wind [4] and vertical axisymmetrical loads act together on the dome. We shall assume that the "preliminary" reinforcement is absent

$$G_k = 0 \text{ for } k \geqslant 3, \quad (4.9)$$

while "additional" refinforcement is laid according to the scheme

$$\psi_1 = \alpha,\ \psi_2 = \beta \text{ for } 0 \leqslant \theta \leqslant \pi,$$
$$\psi_1 = -\alpha,\ \psi_2 = -\beta \text{ for } \pi < \theta < 2\pi, \quad (4.10)$$

where α, β are fixed constants.

From the equations of equilibrium (1.1), under the condition that the forces remain finite at the point $\varphi = 0$, we find

$$T_1 = -\frac{R}{1+\cos\varphi}\left[q_0 + \frac{(q_* - q_0)(1-\cos\varphi)^m}{(m+1)(1-\cos\varphi_*)^m}\right] + T\cos\varphi\cos\theta, \quad (4.11)$$
$$T_2 = -Rq\cos\varphi - Rp\sin\varphi\cos\theta - T_1,$$
$$T_3 = T\sin\theta,\ T(\varphi) = -Rp(2+\cos\varphi)(1-\cos\varphi)^{1/2}[3(1+\cos\varphi)^{3/2}]^{-1}.$$

Using the relations of elasticity (1.2) and the dependences (4.6) and (4.11), for constant c in expression (4.6), we find

$$c = \varepsilon_2(0) = -Rq_0(1-\nu)(2h_0 a\mathrm{E})^{-1},\ h_0 = h|_{\varphi=0}. \quad (4.12)$$

The components of the displacement of the center surface of the dome being examined are determined from the dependences (2.6).

Figures 2a and b show the dependences $H = h/h_0$, G_1, G_2 as a function of φ, θ (for which a strictly moment-free stress state is realized in the dome), calculated using Eqs. (4.3) taking into account (4.4)-(4.12). For definiteness, it is assumed that

$$\theta = 0,\ g(\varphi) = 0,\ \varphi_1 = (3/8)\pi,\ \alpha = 40^\circ,\ \beta = 0^\circ,\ p/q_0 = 0.5,$$
$$q_*/q_0 = 25,\quad m = 3,\quad \varphi_* = (7/16)\pi,\quad \nu = 0.3.$$

Since taking into account (4.10) and (4.11) it can be shown that the functions H, G_1, G_2 are even with respect to $0 \le \theta \le \pi$, it is sufficient to calculate their values for θ. Curves 1-5 in Figs. 2a and b correspond to the values $\theta = 0, \pi/4, \pi/2, (3/4)\pi, \pi$.

For p = 0 we obtain the case when an axisymmetrical vertical load $q(\varphi)$ (4.8) acts on the dome. Then we obtain from (4.3)-(4.6)

$$H = h/h_0 = -\frac{2(T_2 - \nu T_1)}{(1-\nu)Rq_0}\exp\int_0^\varphi\left(1 + \frac{\nu T_2 - T_1}{T_2 - \nu T_1}\right)\operatorname{ctg}\varphi\, d\varphi,\quad G_1 = G_2 = 0,$$

i.e., the dome material must be isotropic, and a moment-free stress state is ensured only due to the change in thickness. The forces T_1, T_2 are determined from Eqs. (4.11) with $p = 0$. The function $H(\varphi)$, calculated in this case, is shown in Fig. 3.

5. Problem 3. Let the following be given: the shape of the center surface, the external load, thickness, rigidity $G_k (k = 4, \ldots, N)$ or preliminary reinforcement, the angles ψ_n $(n = 1, \ldots, N)$, and boundary conditions of the type (1.6)-(1.9). The values of G_1, G_2, G_3 are given on part of the center surface or along separate lines. The problem is to determine the variation, along the entire center surface of the rigidities of the three additional families of filaments with which the stress state in the shell examined will be moment-free.

Assuming, as in problem 2, that T_1, T_2, T_{12} are known from a solution of the equilibrium equations, we represent the relations of elasticity in the form

$$G_1e_1l_{11}^2 + G_2e_2l_{12}^2 + G_3e_3l_{13}^2 = T_1(aEh)^{-1} - a''_{11}\varepsilon_1 - a''_{12}\varepsilon_2 - a''_{13}\varepsilon_3,$$
$$G_1e_1l_{21}^2 + G_2e_2l_{22}^2 + G_3e_3l_{23}^2 = T_2(aEh)^{-1} - a''_{12}\varepsilon_1 - a''_{22}\varepsilon_2 - a''_{23}\varepsilon_3,$$
$$G_1e_1l_{11}l_{21} + G_2e_2l_{12}l_{22} + G_3e_3l_{13}l_{23} = T_{12}(aEh)^{-1} - a''_{13}\varepsilon_1 - a''_{23}\varepsilon_2 - a''_{33}\varepsilon_3,$$

where $e_m = \varepsilon_1 \cos^2\psi_m + \varepsilon_2 \sin^2\psi_m + \varepsilon_3 \sin\psi_m \cos\psi_m$, $m = 1, 2, 3$. From here we find

$$G_l = \Delta_l/\Delta \quad (l = 1, 2, 3), \tag{5.1}$$

where

$$\Delta = -\sin(\psi_1 - \psi_2)\sin(\psi_2 - \psi_3)\sin(\psi_3 - \psi_1); \quad \varepsilon_3 = \varepsilon_{12};$$
$$\Delta_l = \Big\{(aEh)^{-1}[T_1l_{2m}l_{2t} + T_2l_{1m}l_{1t} - T_{12}\sin(\psi_m + \psi_t)] - \sum_{\rho=1}^{3}[a''_{1\rho}l_{2m}l_{2t} +$$
$$+ a''_{2\rho}l_{1m}l_{1t} - a''_{3\rho}\sin(\psi_m + \psi_t)]\,\varepsilon_\rho\Big\}\sin(\psi_m - \psi_t); \quad m, t = 1, 2, 3; \; m \neq l; \; m \neq t.$$

Expressions are obtained for $a''_{11}, a''_{12}, \ldots, a''_{33}$ from Eqs. (4.2) for $a'_{11}, a'_{12}, \ldots, a'_{33}$, if the latter are summed over k beginning with $k = 4$; in the expression $\sin(\psi_m - \psi_t)$ in Δ_l the values for the indices m and t must be chosen in the same order as in the state with indices 1-3 in Δ.

Substituting the values for ε_1, ε_2, ε_3 from (2.1) into (5.1), we obtain expressions for G_1, G_2, G_3 in terms of the functions given according to the condition of the problem and arbitrary functions entering into the general solutions T_1, T_2, T_{12}, ε_1, ε_2, ε_3. Thus the solution of problem 3 reduces to determining the functions of integration from the boundary conditions and the conditions on G_1, G_2, G_3. The displacements are then determined from Eqs. (2.5).

We shall examine the case when the shell is closed in the circular direction and has a smooth vertex and the value of the rigidity G_3 is given on the contour $\theta = \theta_0$:

$$G_3(\varphi, \theta_0) = G(\varphi). \tag{5.2}$$

Then, taking into account the dependence (2.4), we obtain from the expression for G_3 in (5.1)

$$\varepsilon_1 = F - k\varepsilon_2 = F_0 - k_0\varepsilon_2, \tag{5.3}$$

where

$$F_0 = F(\varphi, \theta_0); \quad k_0 = k(\varphi, \theta_0); \quad k = (k_2 + m_2G)\Delta_*; \quad \Delta_* = k_1 + m_1G;$$
$$k_i = \sin(\psi_1 - \psi_3)\sin(\psi_2 - \psi_3)\,l_{i3};$$
$$F = (haE\Delta_*)^{-1}[T_1l_{21}l_{22} + T_2l_{11}l_{12} - T_3\sin(\psi_1 + \psi_2)].$$

From Eqs. (5.3) and (3.4) we find

$$\varepsilon_2 = \exp(-J)\int_0^\varphi \frac{1}{r} F_0R_1\cos\varphi \exp J\,d\varphi, \tag{5.4}$$

$$J = \int_0^\varphi \frac{1 + k_0}{r} R_1 \cos\varphi\, d\varphi.$$

Displacements in the dome are determined from Eqs. (2.6). As an example, we consider a spherical dome with constant thickness, loaded and clamped in accordance with (4.4) and (4.5). We shall assume that

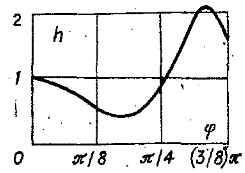

Fig. 3

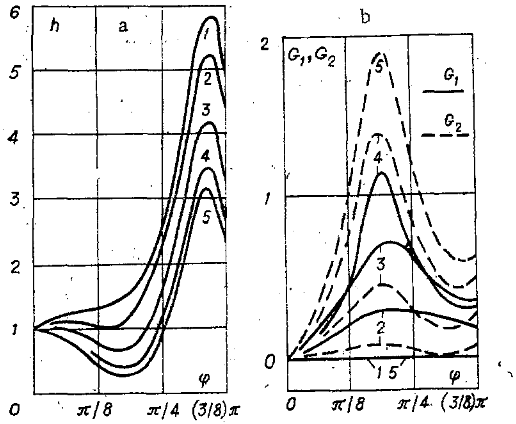

Fig. 2

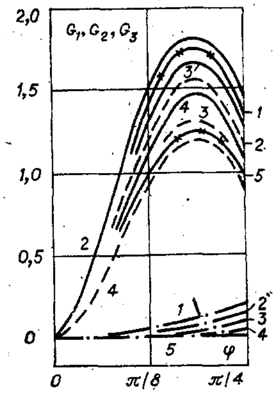

Fig. 4

there is no preliminary reinforcement (N = 3), the magnitudes of the reinforcement angles ψ_1, ψ_2, ψ_3 are constant, and $G(\varphi) = 0$.

The values of G_1 (continuous curves), G_2 (dashed curves), and G_3 (dot–dash curves), for which a moment-free state is realized in the dome under study, calculated from relations (5.1)-(5.4) and (4.11), are presented in Fig. 4. The curves with the cross marks correspond to the case $G_1 = G_2$. In the calculation it was assumed that $\theta_0 = \pi$, $\varphi_1 = \pi/4$, $\varphi_* = (7/16)\pi$, $p/q_0 = 0.5$, $q_*/q_0 = 15$, $m = 1$, $\nu = 0.3$, $\psi_1 = 60°$, $\psi_2 = -60°$, $\psi_3 = 90°$. Curves 1-5 correspond to the values $\theta = 0, \pi/4, \pi/2, (3/4)\pi, \pi$. We also note that the equalities $G_1(\varphi, \theta) = G_2(\varphi, -\theta)$, $G_3(\varphi, \theta) = G_3(\varphi, -\theta)$ are satisfied.

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